ON THE THEORY OF BENDING OF ANISOTROPIC PLATES

(K TEORII IZGIBA ANIZOTROPNYKH PLASTINOK)

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V.V. PONIATOVSKII
(Leningrad)

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The paper uses the method of reference [1] to develop a theory for anisotropic plates, according to which the stresses $\sigma_{\alpha\beta}(\alpha, \beta = 1, 2)$ are expressed in the form of series of Legendre polynomials $P_k(z/h)$.

The results of Reissner [2 and 3] are used in the derivation of the differential equations and boundary conditions. In contrast to other theories of anisotropic plates which do not use Kirchhoff's hypoteses [4 and 5], the present theory makes it possible to take into account more accurately elastic effects at the edge of the plate (edge effects).

Another method for the derivation of the theory of isotropic plates and shallow shells based on the use of series of Legendre polynomials is described in [6].

1. Consider a plate of constant thickness $2h$. We denote the metric tensor of the middle plane of the plate by $g_{\alpha\beta}$ referred to curvilinear coordinates x^α ($\alpha = 1,2$); z is the distance of an arbitrary point in the plate to the middle surface $(-h \leqslant z \leqslant h)$; ∇_α denotes a covariant derivative in the metric on the middle plane. Throughout, Greek indices in tensor notation assume the values 1 and 2. Other indices are enclosed in parentheses, and their position, whether upper or lower, does not alter the meaning of the appropriate symbol.

We assume that the material of the plate has at all points a plane of elastic symmetry parallel to the middle plane.

The strain tensors $e_{\alpha\beta}$, $e_{\alpha z}$, e_{zz} and the stress tensors $\sigma_{\alpha\beta}$, $\sigma_{\alpha z}$, σ_{zz} are related by Hooke's law

$$e_{\alpha\beta} = a_{\alpha\beta\pi\rho}\sigma^{\pi\rho} + a_{\alpha\beta}\sigma_{zz}, \qquad e_{\alpha z} = b_{\alpha\beta}\sigma_z{}^\beta,$$
$$e_{zz} = a_{\alpha\beta}\sigma^{\alpha\beta} + a\sigma_{zz}$$

Here $a_{\alpha\beta\pi\rho}$, $a_{\alpha\beta}$, $b_{\alpha\beta}$ and a are tensors of the strain components. We shall assume that they are independent of z.

We shall now consider a state of stress in the plate which is antisymmetric about the middle plane; analogous results can easily be obtained for the symmetrical case.

We represent the stresses $\sigma_{\alpha\beta}$ in the form of series of Legendre polynomials

$$\sigma_{\alpha\beta} = \sum_{k=1}^{\infty} \sigma_{\alpha\beta}^{(k)} P_{2k-1}(\zeta) \qquad (z = h\zeta \quad -1 \leqslant \zeta \leqslant 1) \tag{1.1}$$

From the equations of equilibrium and the conditions on the planes $z = \pm h$

$$\sigma_{\alpha z} = 0, \qquad \sigma_{zz} = \pm {}^1/_2 p \qquad (z = \pm h)$$

we have the following expressions for the remaining stresses [1]:

$$\sigma_{\alpha z} = h \sum_{k=1}^{\infty} \frac{P_{2k-2}(\zeta) - P_{2k}(z)}{4k-1} \sigma_\alpha^{(k)}$$

$$\sigma_{zz} = h^2 \sum_{k=1}^{\infty} \left[\frac{P_{2k-3}(\zeta)}{(4k-3)(4k-1)} - \frac{2P_{2k-1}(\zeta)}{(4k-3)(4k+1)} + \frac{P_{2k+1}(\zeta)}{(4k-1)(4k+1)} \right] \sigma_{(k)}$$

$$\sigma_{(k)}^{\ \alpha} = \nabla_\beta \sigma_{(k)}^{\ \alpha\beta}, \qquad \sigma_{(k)} = \nabla_\alpha \sigma_{(k)}^{\ \alpha}; \qquad P_k(\zeta) \equiv 0 \qquad (k < 0) \tag{1.2}$$

and an equation equivalent to the equilibrium equation in terms of stresses

$${}^2/_3 h^2 \nabla_\alpha \sigma_{(1)}^{\ \alpha} + p = 0 \tag{1.3}$$

We introduce the infinite matrices

$$D = \| d_{ik} \|, \quad D = ID \quad d_{ik} = \frac{\delta_{ik}}{4i-1}, \qquad \delta_{ik} = \begin{cases} 0 & (i \neq k) \\ 1 & (k = i) \end{cases}$$

$$A = \| a_{ik} \|, \qquad A' = IA, \qquad a_{ik} = -\frac{\delta_{i-1,k}}{(4i-5)(4i-3)(4i-1)} +$$

$$+ \frac{2\delta_{ik}}{(4i-3)(4i-1)(4i+1)} - \frac{\delta_{i,k-1}}{(4i-1)(4i+1)(4i+3)} \qquad (i, k = 1, 2, 3, \ldots)$$

$$B = \| b_{ik} \|, \qquad b_{1k} = 0, \qquad b_{ik} = \frac{\delta_{i-2,k}}{(4i-9)(4i-7)(4i-5)(4i-3)(4i-1)} -$$

$$- \frac{4\delta_{i-1,k}}{(4i-7)(4i-5)(4i-3)(4i-1)(4i+1)} +$$

$$+ \frac{6\delta_{ik}}{(4i-5)(4i-3)(4i-1)(4i+1)(4i+3)} -$$

$$- \frac{4\delta_{i,k-1}}{(4i-3)(4i-1)(4i+1)(4i+3)(4i+5)} +$$

$$+ \frac{\delta_{i,k-2}}{(4i-1)(4i+1)(4i+3)(4i+5)(4i+7)} \qquad \begin{pmatrix} i = 2, 3, \ldots \\ k = 1, 2, \ldots \end{pmatrix}$$

Here I is an infinite diagonal matrix in which all the elements of the leading diagonal are unity except the first, which is zero.

In addition, we introduce the vectors $\sigma_{\alpha\beta}$, σ_α and σ defined by

$$\sigma_{\alpha\beta} = (\sigma_{\alpha\beta}^{(1)}, \sigma_{\alpha\beta}^{(2)}, \ldots), \qquad \sigma_\alpha = (\sigma_\alpha^{(1)}, \sigma_\alpha^{(2)}, \ldots), \qquad \sigma = (\sigma_{(1)}, \sigma_{(2)}, \ldots)$$

so that from (1.2) we have

$$\sigma^\alpha = \nabla_\beta \sigma^{\alpha\beta}, \qquad \sigma = \nabla_\alpha \sigma^\alpha$$

Proceeding as in [1], we obtain Equations

$$
\begin{aligned}
{}^1/_3 a_{\alpha\beta\pi\rho}\sigma_{(1)}^{\pi\rho} - {}^2/_{15}h^2\,[\nabla_\alpha\,(b_{\pi\beta}\sigma_{(1)}^{\pi}) + \nabla_\beta\,(b_{\pi\alpha}\sigma_{(1)}^{\pi})] + {}^1/_{105}h^2\,[\nabla_\alpha\,(b_{\pi\beta}\sigma_{(2)}^{\pi}) + \\
+ \nabla_\beta\,(b_{\pi\alpha}\,\sigma_{(2)}^{\pi})] + {}^1/_{105}h^2 a_{\alpha\beta}\sigma_{(2)} + {}^1/_3 h\nabla_\alpha\nabla_\beta w = -{}^1/_5 p a_{\alpha\beta} \qquad (1.4) \\
a_{\alpha\beta\pi\rho}D'\sigma^{\pi\rho} - h^2A'\,[a_{\alpha\beta}\sigma + \nabla_\alpha\nabla_\beta\,(a_{\pi\rho}\sigma^{\pi\rho}) + \\
+ \nabla_\alpha\,(b_{\pi\beta}\sigma^{\pi}) + \nabla_\beta\,(b_{\pi\alpha}\sigma^{\pi})] + h^4B\nabla_\alpha\nabla_\beta\,(a\sigma) = 0
\end{aligned}
$$

and the homogeneous geometrical boundary conditions

$$
\left(\frac{4h^2b_{\alpha\beta}\sigma_{(1)}^{\beta}}{15} - \frac{2h^2b_{\alpha\beta}\sigma_{(2)}^{\beta}}{105}\right)n^\alpha - \frac{h}{3}\frac{\partial w}{\partial x^\alpha}n^\alpha = 0 \quad (n^\alpha \sim s^\alpha) \qquad w = 0 \qquad (1.5)
$$

$$
2h^2n^\alpha b_{\alpha\beta}A'\sigma^\beta - h^2n^\alpha A'\frac{\partial a_{\pi\rho}\sigma^{\pi\rho}}{\partial x^\alpha} + h^4n^\alpha B\frac{\partial a\sigma}{\partial x^\alpha} = 0 \qquad (n^\alpha \sim s^\alpha)
$$

$$
h^2aB\sigma - a_{\alpha\beta}A'\sigma^{\alpha\beta} = 0
$$

The corresponding static boundary conditions are

$$
\sigma_{\alpha\beta}n^\alpha n^\beta = \sigma_{nn}, \qquad \sigma_{\alpha\beta}n^\alpha s^\beta = \sigma_{ns}, \qquad \sigma_\alpha n^\alpha = \sigma_n \qquad (1.6)
$$

Here n^α are the contravariant components of the vector of the unit external normal to the boundary of the middle plane of the plate, s^α are the components of the unit tangential vector. A scalar function w is introduced as in [1], and represents the characteristic deflection of the planes $z=\pm h$. The symbol $(n^\alpha \sim s^\alpha)$ indicates that there exist relations which can be obtained from those quoted by replacing n^α by s^α.

2. Consider a homogeneous transversely isotropic plate the plane of isotropy of which coincides with the middle plane. E, E_z, ν, ν_z denote the Young's moduli and Poisson's ratios for directions in the plane of isotropy and for directions perpendicular to this plane; G represents the shear modulus for planes perpendicular to the plane of isotropy. The tensors of strain coefficients have the form

$$
a_{\alpha\beta\pi\rho} = \frac{1}{E}\,[(1+\nu)\,g_{\alpha\pi}g_{\beta\rho} - \nu g_{\alpha\beta}g_{\pi\rho}]
$$

$$
a_{\alpha\beta} = -\frac{\nu_z}{E_z}g_{\alpha\beta}, \qquad b_{\alpha\beta} = \frac{1}{2G}g_{\alpha\beta}, \qquad a = \frac{1}{E_z} \qquad (2.1)
$$

We introduce functions of the stresses $\omega = (\omega_{(1)}, \omega_{(2)}, \ldots)$ and $\Phi = (\Phi_{(1)}, \Phi_{(2)}, \ldots)$. For this purpose we resolve the vector σ_α into a potential and a rotational part, setting

$$
\begin{aligned}
\sigma_\alpha = t_\alpha + \tau_\alpha, \qquad t_\alpha = \nabla_\alpha\Phi, \qquad \tau_\alpha = \varepsilon_{\cdot\alpha}{}^{\beta}\nabla_\beta\omega \\
t_\alpha = (t_\alpha^{(1)}, t_\alpha^{(2)}, \ldots), \qquad \tau_\alpha = (\tau_\alpha^{(1)}, \tau_\alpha^{(2)}, \ldots)
\end{aligned} \qquad (2.2)
$$

Here $\varepsilon_{\cdot\alpha}{}^{\beta}$ is the mixed form of the discriminant

$$
\varepsilon_{\alpha\alpha} = 0, \qquad \varepsilon_{12} = -\varepsilon_{21} = \sqrt{g} = \sqrt{g_{11}g_{22} - g_{12}^2}
$$

Substituting (2.1) and (2.2) into (1.3) and (1.4), we obtain Equations

$$
{}^2/_3 h^2\Delta\Phi_{(1)} + p = 0 \qquad (2.3)
$$

$$
\begin{aligned}
&{}^1/_3\,[(1+\nu)(\sigma_{\alpha\beta}^{(1)}-\tau_{\alpha\beta}^{(1)})-\nu g_{\alpha\beta}\sigma_{\pi(1)}^{\pi}]+{}^1/_3Eh\,\nabla_\alpha\nabla_\beta w-\\
&-{}^2/_{15}h^2EG^{-1}\nabla_\alpha\nabla_\beta\Phi_{(1)}+{}^1/_{105}h^2EG^{-1}\nabla_\alpha\nabla_\beta\Phi_{(2)}-\\
&-{}^1/_{105}\nu_z h^2EE_z^{-1}g_{\alpha\beta}\Delta\Phi_{(2)}-{}^1/_5\nu_zEE_z^{-1}g_{\alpha\beta}p=0 \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
&(1+\nu)D'(\sigma_{\alpha\beta}-\tau_{\alpha\beta})-\nu g_{\alpha\beta}D'\sigma_\pi^\pi-h^2EG^{-1}A'\nabla_\alpha\nabla_\beta\Phi+\\
&+h^2E\nu_zE_z^{-1}A'(g_{\alpha\beta}\Delta\Phi+\nabla_\alpha\nabla_\beta\sigma_\pi^\pi)+Eh^4E_z^{-1}B\nabla_\alpha\nabla_\beta\Delta\Phi=0\\
&(\Delta=g^{\alpha\beta}\nabla_\alpha\nabla_\beta)
\end{aligned}
$$

where $\tau_{\alpha\beta}=(\tau_{\alpha\beta}^{(1)},\tau_{\alpha\beta}^{(2)},\ldots)$ denotes the tensor

$$D\tau_{\alpha\beta}=\frac{h^2E}{2(1+\nu)G}\nabla_\pi(\varepsilon_{\cdot\beta}^{\pi}\nabla_\alpha A\omega+\varepsilon_{\cdot\alpha}^{\pi}\nabla_\beta A\omega) \qquad (2.5)$$

We impose the requirement that the tensor (2.5) satisfies the relation

$$\nabla_\beta\tau^{\alpha\beta}=\tau^\alpha=\varepsilon^{\beta\alpha}\nabla_\beta\omega$$

We then obtain the following relation for determining ω :

$$\left(D-\frac{Eh^2\Delta}{2(1+\nu)G}A\right)\omega=0 \qquad (2.6)$$

We now set $\sigma_{\alpha\beta}-\tau_{\alpha\beta}=t_{\alpha\beta}$. From (2.5) and (2.2) we have that

$$\sigma_\alpha{}^\alpha=t_{\alpha\alpha},\qquad \nabla_\beta t^{\alpha\beta}=t^\alpha$$

Consequently, of the unknown functions, Equations (2.3) and (2.4) contain only $t_{\alpha\beta}$ and w .

Thus the tensor $\sigma_{\alpha\beta}$ is expressed as the sum of two tensors, $\tau_{\alpha\beta}$ and $t_{\alpha\beta}$. The former can be determined from Formula (2.5), in which the vector ω must satisfy Equation (2.6) and has the property that $\tau_\alpha{}^\alpha=0$ and that the vector $\nabla_\beta\tau^{\alpha\beta}$ is solenoidal. The second tensor $t_{\alpha\beta}$ is characterized only by the fact that its divergence $\nabla_\beta t^{\alpha\beta}$ is a potential vector.

In order to express the tensor $t_{\alpha\beta}$ in terms of Φ and w and obtain equations for finding these functions, we take the contravariant derivative of (2.4) with the subsequent contraction. As a result we obtain

$$
\begin{aligned}
&\frac{1+\nu}{3}\Phi_{(1)}-\frac{\nu}{3}t_{\alpha(1)}^{\alpha}+\frac{Eh}{3}\Delta w-\frac{2h^2}{15}\left(\frac{E}{G}-\frac{E\nu_z}{E_z}\right)\Delta\Phi_{(1)}+\\
&\qquad+\frac{h^2}{105}\left(\frac{E}{G}-\frac{E\nu_z}{E_z}\right)\Delta\Phi_{(2)}=0 \qquad (2.7)
\end{aligned}
$$

$$(1+\nu)D'\Phi-\nu D't_\alpha{}^\alpha-h^2\left(\frac{E}{G}-\frac{E\nu_z}{E_z}\right)A'\Delta\Phi+\frac{h^2E\nu_z}{E_z}A'\Delta t_\alpha{}^\alpha+\frac{Eh^4}{E_z}B\Delta\Delta\Phi=0$$

Subtracting from these equations the result of the contraction of (2.4) with the metric tensor, we obtain

$$Dt_\alpha{}^\alpha=(1+\nu)D\Phi-h^2EE_z{}^{-1}\nu_z\Delta A\Phi$$

Eliminating $t_\alpha{}^\alpha$ from (2.7) and (2.4) with the aid of this relation we find that

$$\frac{1-\nu^2}{3}\Phi_{(1)}-\frac{2h^2}{15}\left(\frac{E}{G}-\frac{E\nu_z(1+\nu)}{E_z}\right)\Delta\Phi_{(1)}+ \tag{2.8}$$
$$+\frac{h^3}{15}\left(\frac{E}{G}-\frac{E\nu_z(1+\nu)}{E_z}\right)\Delta\Phi_{(2)}+\frac{Eh}{3}\Delta w=0$$

$$\Big[(1-\nu^2)D'-\left(\frac{E}{G}-2\nu_z(1+\nu)\frac{E}{E_z}\right)h^2\Delta A'+ \tag{2.9}$$
$$+\frac{E}{E_z}\left(1-\frac{E}{E_z}\nu_z^2\right)h^4\Delta\Delta B\Big]\Phi=0$$

$$\frac{1+\nu}{3}t_{\alpha\beta}^{(1)}=-\frac{Eh}{3}\nabla_\alpha\nabla_\beta w+\frac{E\nu_z(1+\nu)}{5E_z}g_{\alpha\beta}p+\frac{\nu(1+\nu)}{3}g_{\alpha\beta}\Phi_{(1)}+$$
$$+\frac{2h^2E}{15G}\nabla_\alpha\nabla_\beta\Phi_{(1)}-\frac{h^3E}{105C}\nabla_\alpha\nabla_\beta\Phi_{(2)}+\frac{h^2E\nu_z(1+\nu)}{105E_z}g_{\alpha\beta}\Delta\Phi_{(2)} \tag{2.10}$$

$$-(1+\nu)D't_{\alpha\beta}=\Big[-\nu(1+\nu)g_{\alpha\beta}D'+\frac{E}{E_z}\nu_z(1+\nu)h^2\varepsilon_{.\alpha}^{\pi}\varepsilon_{.\beta}^{\rho}\nabla_\pi\nabla_\rho A'-$$
$$-\left(\frac{E}{G}-2\nu_z(1+\nu)\frac{E}{E_z}\right)h^2\nabla_\alpha\nabla_\beta A'+\frac{E}{E_z}\left(1-\frac{E}{E_z}\nu_z^2\right)h^4\nabla_\alpha\nabla_\beta\Delta B\Big]\ \Phi \tag{2.11}$$

Equations (2.8), (2.9) and (2.3) from a system of equations for determining w and Φ, and Formulas (2.10) and (2.11) define $t_{\alpha\beta}$ in terms of these functions.

3. We shall solve the problem by the method of asymptotic integration of equations having derivatives multiplied by a small parameter [7 and 8]; the half thickness of the plate h is assumed to be small compared with the characteristic linear dimension a of the middle plane. It is assumed that the parameters of the problem and the required solution are sufficiently smooth functions of a point on the middle plane.

We seek the integrals of the basic state of stress, which do not vary rapidly (the regular terms of the asymptote), in the form of the expansions

$$\sigma_{\alpha\beta}^{(k)}=h^{-2}(\sigma_{\alpha\beta}^{(k0)}+h\sigma_{\alpha\beta}^{(k1)}+\ldots),\qquad w=h^{-3}(w^{(0)}+hw^{(1)}+\ldots) \tag{3.1}$$

If we now substitute (3.1) into Equations (2.3) and (2.4) and equate to zero the sum of terms with equal powers of h, we obtain a recurrent sequence of equations for determining the functions $\sigma_{\alpha\beta}^{(ks)}$, $w^{(s)}$. These equations reduce to nonhomogeneous biharmonic equations and in a first approximation coincide with the equations of the theory.

In order to obtain integrals of the boundary-layer type (ordinary edge-effects) we introduce a local orthogonal system of coordinates $x^1=r$, $x^2=s$ in the neighborhood of the boundary Γ of the region occupied by the middle plane, where r is the distance from points on the curve Γ along the external normal and s is the arc-length along the curve Γ.

Returnig to Equations (2.6), let us take D and A to represent truncated matrices of order m, and w to represent an m-dimensional vector.

We expand the Laplacian operator [7] in the neighborhood of the boundary Γ in powers of h

$$h^2\Delta = \frac{\partial^2}{\partial t^2} + \frac{h}{R}\frac{\partial}{\partial t} + h^2\left(\frac{\partial^2}{\partial s^2} - \frac{t}{R^2}\frac{\partial}{\partial t}\right) + \ldots \qquad (r = ht) \qquad (3.2)$$

Here R is the radius of curvature of the curve Γ.

We seek the vector ω of the boundary-layer type in the form of the expansion

$$\omega = h^a\left(\omega^{(0)} + h\omega^{(1)} + \ldots\right) \qquad (3.3)$$

(a is an integer). If substitute (3.2) and (3.3) into Equation (2.6) and equate to zero the sum of equal powers of h, we obtain a recurrent sequence of ordinary linear differential equations with constant coefficients (s appears as a parameter). It is required of the solutions to these equations that they are of the boundary-layer type, and to obtain them we use the positive roots of the characteristic equation

$$\left| D - \frac{E\lambda^2}{2(1+\nu)G} A \right| = 0$$

This equation has m positive roots so that the integrals (3.3) of the boundary-layer type have m degrees of freedom at the boundary (for $t = 0$).

The solution of the boundary-layer type to the homogeneous equations (2.3) and (2.9) (with $p = 0$) is found in an analogous way

$$\Phi_{(1)} = 0, \quad \Phi_{(k)} = h^b\left({\Phi_{(k)}}^{(0)} + h{\Phi_{(k)}}^{(1)} + \ldots\right) \qquad (k = 2, 3, \ldots, m) \qquad (3.4)$$

Here b is an integer.

It can be shown that integrals (3.4) of the boundary-layer type have $2m - 2$ degrees of freedom at the boundary (for $t = 0$), i.e. the corresponding characteristic equation has $2m - 2$ roots with positive real parts.

Having determined the functions $\Phi_{(k)}$ of the boundary-layer type we find from (2.8) and (2.10) the boundary-layer functions

$$w = \frac{h}{35}\left(\frac{\nu_z(1+\nu)}{E_z} - \frac{1}{G}\right)\Phi_{(2)}, \qquad t_{\alpha\beta}^{(1)} = \frac{E\nu_z}{E_z}\frac{h^2}{35}\varepsilon^{\pi}_{\cdot\alpha}\varepsilon^{\rho}_{\cdot\beta}\nabla_\pi\nabla_\rho\Phi_{(2)}$$

The boundary-layer functions $\tau_{\alpha\beta}$ and $t_{\alpha\beta}^{(k)}$ $(k = 2, 3, \ldots, m)$ can be found from Formulas (2.5) and (2.11).

The regular terms of the asymptote in each approximation can be subjected to two boundary conditions; $3m - 2$ boundary conditions can be satisfied with the aid of integrals of the boundary-layer functions (3.3) and (3.4). Thus, by means of the integrals obtained we can satisfy the $3m$ boundary conditions which are obtained from (1.5) and (1.6) by retaining in the series (1.1) the first m terms.

The boundary conditions must be satisfied by a process of superposition [8]. The regular terms of the asymptote and the boundary-layer functions are substituted in the boundary conditions of the problem and then the integers a and b (appearing in (3.3) and (3.4)) are selected in an appropriate manner. In this way a recurrent system of linear algebraic equations can be obtained for finding the constants of integration in (3.3) and (3.4) and the boundary conditions can be specified for subsequently determining the regular terms of the asymptote. The latter, which determine the basic state of stress in the plate, and the boundary-layer functions, which determine the edge effects, are interrelated: the latter are determined by the former and the behavior of the boundary-layer functions $\omega_{(1)}, \omega_{(2)}, \Phi_{(2)}$ at the boundary determines the boundary conditions for the regular terms of the asymptote.

The asymptotic investigation of the state of stress in a plate leads to conclusions obtained by other methods [9 to 12] for isotropic plates. Outside a narrow edge zone the error in Kirchhoff's hypotheses is of the order of h compared with a – the characteristic dimension of the middle plane. Close to the boundary the shear stresses and the stresses normal to the middle plane, namely $\sigma_{\alpha z}$ and σ_{zz} which are ignored in the classical theory, are of the same order as $\sigma_{\alpha\beta}$. Therefore the error in Kirchhoff's hypotheses near the boundary for sufficiently small h can be considerable.

4. In the theory of anisotropic plates by Ambartsumian [4 and 5] and also in the theory by Reissner [2 and 3] one edge effect $\omega_{(1)}$ is taken into account. In contrast to these theories the present study enables a solution to be found with any finite number of edge effects.

The ordinary linear differential equations, to the solution of which one reduces the derivation of the boundary-layer functions, are independent of the geomerty of the plate and can be integrated once and for all. For purposes of practical computation it is sufficient to solve these equations for small values of m since the boudary-layer functions $\omega_{(k)}$ and $\Phi_{(k)}$ for large k are rapidly damped and their effect on the basic state of stress is considerably reduced.

An investigation of problems of bending of plates for various support conditions (full fixity, simple support) shows that in the second and third approximations the effect of the boundary-layer functions $\Phi_{(k)}$ on the basic state of stress is only slight. It disappears completely if Poisson's ratio ν_z is zero. For example, for a circular plate of radius a under a uniform load of intensity p and with built-in edges we have as a second approximation for the deflection function

$$w = \frac{1-\nu^2}{Eh^3}\frac{3p}{128}(r^2-a^2)\,[(r^2-a^2)+ahC] \qquad (C = \text{const})$$

where r is the distance from the centre of the circle.

The second term in the square brackets applies the correction to the classical theory which results from taking into account the boundary-layer function $\Phi_{(2)}$. For an isotropic plate and for $\nu = 0.3$ we have that $C \approx 0.05$. For an anisotropic plate with material properties $E/E_z = 7/5$, $\nu_z = 0.5$, $\nu = 0$, $E/G = 0.7$, $C \approx 0.19$.

Let us take one further example. Consider a rectangular plate $(a \times b)$ under the action of a load given by

$$p = q \sin(\pi x / a) \sin(\pi y / b)$$

Here q is the load intensity at the centre $(x = \frac{1}{2}a,\ y = \frac{1}{2}b)$. The plate is freely supported at the edges. At $x = 0$ we have that $\sigma_{xy} = 0$ and the remainder of the boundary is fixed against displacements tangential to the contour.

Here, in the second approximation only $\omega_{(k)}$ has an effect on the basic state of stress. We obtain the following expression for the deflection function w :

$$w = w_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + w_0 A(1-\nu)\frac{\pi^2 h}{a}\left(\frac{E}{2(1+\nu)G}\right)^{1/2}\sin\frac{\pi y}{b}\left[\frac{x}{a}\coth\frac{\pi a}{b}\cosh\frac{\pi x}{b} - \right.$$
$$\left. - \frac{a\sinh(\pi x/b)}{b\sinh^2(\pi a/b)} - \frac{x}{b}\sinh\frac{\pi x}{b}\right]$$

Here w_0 is the deflection at the center of the place as given by the classical theory and A is a constant. The second term, which depends on the ratio h/a, gives a correction to the classical theory. The numerical coefficient A has the value 0.631 if the edge effects are taken into account ($m = 2$). According to theories [3 to 5] which consider one edge effect $A = \sqrt{0.4} \approx 0.6325$. Thus the second edge effect introduces only a minor correction to the value of the constant A. The correction of the basic state of stress which in the other problems results from taking into account the boundary-layer functions $\Phi_{(k)}$, is insignificant and theories [3 to 5] can therefore be applied to obtain the first correction to the basic state of stress as given by the classical theory. The magnitude of this correction increases with increase in the ratio E/G and can become significant for strongly anisotropic plates with large values of E/G

However, theories [3 to 5] cannot give a correct representation of the state of stress in a plate close to its edges mainly because they take no account of any stress state associated with the boundary-layer function $\Phi_{(k)}$.

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